

Holographic dual of Cold Trapped Fermions

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Abstract

We study cold fermionic atoms using the holographic principle. We note that current atomic experiments with massive fermions trapped in a harmonic potential in the unitarity limit behave as massless fermions thanks to the Thomas-Fermi approximation. We map the thermodynamics of strongly correlated massless fermion to that of the charged black hole and study the thermodynamics and transport properties of cold fermions at strong coupling at finite temperature and density. In cold limit, the specific heat of charged black hole is linear in T independent of the dimensionality, which is reminiscent of Fermi liquids. The shear viscosity per particle is shown to be finite as a consequence of the non-vanishing of the entropy. We show that our holographic results compare favorably with most of the current atomic data.

1 Introduction

Recent atomic measurements involving trapped cold fermions have unravelled a wealth of results regarding the behavior of dilute but strongly interacting fermionic systems [1]. By varying the strength of the magnetic trap, atoms were driven through the notorious Feshbach resonance making their scattering length for all purposes infinite.

Strongly interacting cold fermions with an infinite scattering length exhibit universal bulk and transport properties. In particular, the ground state energy per particle was found to asymptote $E/NE_F \approx 0.5$ independently of the details of the interaction [2]. In the long wavelength limit, strongly interacting fermions behave hydrodynamically for a broad range of temperatures above the superfluid temperature [3].

It was suggested in [4] that cold fermionic systems may exhibit also universal transport properties with in particular a quantum bound on the viscosity to density ratio with $\eta/n \geq \hbar/6\pi$. This bound is analogous but distinct from the quantum bound for hot gauge theories established using the holographic principle [5]. Indeed, transport in cold systems is affected by the interactions through possibly a Fermi surface or a fermionic crystal as opposed to transport in hot systems which is the result of rescattering in the heat bath. The cold quantum bound on the viscosity was recently explored experimentally in [6].

Finite fermion number as well as temperature is crucial to analyze strongly coupled fermionic systems in the context of the gravity dual theories. Finite fermion density in the holographic approach was introduced in [7] and has since been discussed by many. The gravity back reaction of the dense matter is also encoded by considering bulk filling branes [9], which allows the charge of the RN AdS black hole to be identified with the fermionic charge. The hydrodynamics and transport coefficients of this system have been analysed in [10, 11], where fermionic corrections at high temperature have been discussed.

In this paper, we suggest that the RN AdS black hole can describe cold fermions as it interpolates continuously between hot and cold massless fermions. We use dual gravity to study the bulk thermodynamics and transport near the cold limit. In the cold limit, the specific heat of charged black hole is linear in T independent of dimension like in Fermi liquids. Notice that this is consistent with recent studies of fermionic propagators in [12, 13, 14] which report the the presence of the fermi surface at zero temperature. We also note that recently in [15, 16] a non-relativistic gravity background was suggested for cold atoms. Here we use the relativistic gravity background based on the observation that nonrelativistic atoms in harmonic traps behave similarly to massless fermions in the Thomas-Fermi approximation. The latter is well justified for current atomic traps with about a million atoms at the coldest

temperature.

In section 2, we briefly review the holographic model. In section 3 we detail the thermodynamical results at finite temperature and density. In section 4 we analyze the cold limits of transport properties and derive the longitudinal and transverse fermionic conductivities. In section 5, we interpret the results for zero temperature using the visco-elastic framework to derive the constitutive equations for the longitudinal and transverse fermionic currents. In section 6, we show that magnetically trapped massive fermions behave as free massless fermions in 3 spatial dimensions. In section 7, we compare our results to a number of thermodynamic and transport results established recently using cold and trapped fermions. We conclude with summary, a caution and prospects in section 8.

2 Holographic Model

To analyze cold fermions at low temperature we make use of a simple holographic model [9] in which fermions are included as a U(1) charge source on the bulk filling branes and the fermion back reaction on the metric is included. Although the original model was motivated from the D3-D7 embedding, here we introduce it as a toy model describing the condensed matter system at the boundary. The model is the Reissner-Nordstrom AdS (RN-AdS) black hole metric [8] coupled to a U(1) flavor bulk field sourced by the fermion charge on the BH. We take the RN AdS BH as a model in the bottom up approach. The global fermionic charge should be identified with a local U(1) gauge field in bulk. As we have only one U(1) vector gauge field in bulk, it is natural to identify it as the dual to the fermionic charge on the boundary in our bottom up approach.

The thermodynamics [8] and transport [10, 11] in the RN-AdS have been analyzed with a particular emphasis on density corrections to the high temperature limit [9]. Here instead, we will be mostly interested in the cold temperature limit of the RN-AdS results as they may shed light on bulk thermodynamics and transport in cold and non-confining fermionic systems at strong coupling as we detail below. The interest in these systems is obvious from the wealth of atomic experiments that have sprung in the past decade using atomic trap experiments with cold fermions near the Feshbach resonance.

The effective action describing bulk RN-AdS gravity sourcing a U(1) gauge field reads

$$S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} (R - 2\Lambda) - \frac{1}{4e^2} \int d^5x \sqrt{-g} F^2 \quad (1)$$

with $\kappa^2 = 8\pi G_5$ and $\Lambda = -6/l^2$ are the gravitational and cosmological constant¹. At this stage, one may wonder how to map the gravity parameters to a system of cold fermions. Our strategy is to express all the bulk quantities in terms of κ and the ratio $\gamma := \kappa/le$. As we show below, κ drops in the thermodynamic *per particle* and γ is mapped on a universal parameter of the cold fermionic systems.

The metric for the RN-AdS black-hole is

$$ds^2 = \frac{r^2}{l^2} (-f dt^2 + d\vec{x}^2) + \frac{l^2}{r^2 f} dr^2 \quad (2)$$

and the U(1) gauge field is

$$A_4 = \mu - \frac{Q}{r^2} \quad (3)$$

with $f = 1 - ml^2/r^4 + q^2 l^2/r^6$. The gravitational equations of motion tie the electric charge Q to the geometrical charge q through

$$\frac{q^2}{Q^2} = \frac{2}{3} \frac{\kappa^2}{e^2} \quad (4)$$

The position of the RN-AdS black-hole horizon is fixed by the maximum zero of $f(r_+) = 0$, which exists if and only if the geometrical charge q satisfies the inequality $q^4 \leq 4m^3 l^2/27$. Again m and l are the mass and length of the RN-AdS black hole space, with r_+ bounded by $l\sqrt{m/3} \leq r_+^2 \leq l\sqrt{m}$. The fermion chemical potential μ in (3) is fixed by demanding that $A_4(r_+) = 0$ on the black-hole horizon. This latter condition enforces the Gibbs relations. The temperature in the RN-AdS space is defined by the singularity free conical condition

$$T = \frac{r_+^2 f'(r_+)}{4\pi l^2} = 1/\beta \quad (5)$$

The regulated Gibbs energy $\Omega = \Delta S/\beta$ follows from (1) by inserting the RN-AdS charged black hole (2-3) and subtracting the *empty* thermal AdS contribution [9]. The result is

$$\Omega = -\frac{V_3}{2\kappa^2 l^3} \left(\frac{r_+^4}{l^2} + \frac{q^2}{r_+^2} \right) \quad (6)$$

by trading $m = r_+^4/l^2 + q^2/r_+^2$. The densities of entropy s , energy ϵ and pressure p follow

¹ If we consider this model as stemming from a leading order approximation of the D3-D7 embedding, the U(1) gauge coupling is $l/e^2 = N_c N_f/(2\pi)^2$ and $l^3/\kappa^2 = N_c^2/4\pi^2$. In 10 dimensions the bulk filling procedure is optimal for D9 rather than D7 branes. For D9, $1/e^2 = (N_c N_f/32\pi^3)l/l_s^2$ and $l^3/\kappa^2 = N_c^2/4\pi^2$.

from (6) through the usual grand-canonical rule [8, 9]

$$\begin{aligned}s &= \frac{2\pi r_+^3}{\kappa^2 l^3} = \frac{\pi l^3}{4\kappa^2 b^3} \\ \epsilon &= \frac{3m}{2\kappa^2 l^3} = \frac{3l^3}{32\kappa^2 b^4}(1+a) = 3p\end{aligned}\tag{7}$$

with $a = q^2 l^2 / r_+^2$ and $b = l^2 / 2r_+$. The fermion chemical potential and number density are $\mu = Q/r_+^2$ and $n = 2Q/e^2 l^3$.

3 Thermodynamics at Low Temperature

The above thermodynamical relations hold for all values of temperature T and chemical potential μ for fixed $\gamma = \kappa/le$. First, we examine the thermodynamics in the cold limit to acquire more physical insights on the cold system. The fermions that source the U(1) vector fields are expected to be in the Coulomb phase with strong vector interactions. At weak coupling the particle-hole (ph) interaction is larger than the particle-particle (pp) interaction. However, small gaps and phase space rules in favor of the BCS pairing over the Overhauser mechanism [17]. What happens at strong coupling is currently speculative, in the absence of first principle calculations. The current holographic model may provide some insights to this question.

Unwinding the above thermodynamical relations in terms of temperature and chemical potential yield for the Gibbs potential

$$\Omega/V_3 = \mu n = \epsilon + p - Ts\tag{8}$$

Introducing the parameter N_c through

$$N_c := \frac{2\pi l^{3/2}}{\kappa},\tag{9}$$

the fermion number density is explicitly

$$n = \frac{N_c^2 \gamma^2}{8} \mu T^2 \left(1 + \sqrt{1 + \frac{4\gamma^2 \mu^2}{3\pi^2 T^2}} \right)^2\tag{10}$$

with an expansion around the cold point of the form

$$n = \frac{N_c^2 \gamma^4}{6\pi^2} \mu^3 + \frac{N_c^2 \gamma^3}{2\pi\sqrt{3}} \mu^2 T + \frac{1}{4} N_c^2 \gamma^2 \mu T^2 + \frac{\sqrt{3}\pi}{16} N_c^2 \gamma T^3 + \mathcal{O}(T^5) \quad (11)$$

Note that the presence of linear term in (11) can be readily seen from (10) when T^2 is combined in the squared bracket, ie

$$\left(T + \sqrt{T^2 + \frac{4\gamma^2 \mu^2}{3\pi^2}} \right)^2.$$

The RN-AdS black-hole at zero temperature is characterized by a finite fermion density which is proportional to $N_c^2 \gamma^4$. This result is the analogue of the R-charge density in the D3 case with a U(1) vector potential, ie $n = N_c^2 \mu^3 / 96\pi^2$ with $\gamma = 1/2$. The small temperature correction is linear in T and of order $N_c^2 \gamma^3$.

The energy density reads

$$\epsilon = \frac{3\pi^2 N_c^2 T^4}{128} \left(1 + \sqrt{1 + \frac{4\gamma^2 \mu^2}{3\pi^2 T^2}} \right)^4 \left(1 + \frac{8N_f \mu^2}{3N_c \pi^2 T^2} \left(1 + \sqrt{1 + \frac{4\gamma^2 \mu^2}{3\pi^2 T^2}} \right)^{-2} \right) \quad (12)$$

and its expansion around the cold fermionic point is

$$\epsilon = \frac{N_c^2 \gamma^4}{8\pi^2} \mu^4 + \frac{N_c^2 \gamma^3}{2\pi\sqrt{3}} \mu^3 T + \frac{3N_c^2 \gamma^2}{8} \mu^2 T^2 + \mathcal{O}((T/\mu)^3) \quad (13)$$

Note that the equation of state at zero temperature

$$\epsilon = \frac{3}{4} \left(\frac{6\pi^2}{\gamma^2 N_c^2} \right)^{1/3} n^{4/3} \quad (14)$$

suggest that this is a system of degenerate massless fermions. This dual interpretation is supported by transport properties as we discuss below. These massless states are the analogue to the BPS marginal bound states in the supersymmetric theory.

A similar take on the current results follows from the entropy through

$$s = \frac{\pi^2 N_c^2 T^3}{16} \left(1 + \sqrt{1 + \frac{4\gamma^2 \mu^2}{3\pi^2 T^2}} \right)^3 \quad (15)$$

and its expansion around the cold point

$$s = \frac{N_c^2 \gamma^3}{6\pi\sqrt{3}} \mu^3 + \frac{N_c^2 \gamma^2}{4} \mu^2 T + \frac{3\sqrt{3}\pi N_c^2 \gamma}{16} \mu T^2 + \mathcal{O}(T/\mu)^3 \quad (16)$$

The entropy density is finite at zero temperature and proportional to $N_c^2 \gamma^3 \mu^3$. This cold entropy is the degree of degeneracy of the fermionic liquid.

The specific heat near the cold point is

$$c_V = T \frac{ds}{dT} = \frac{N_c^2 \gamma^2}{4} \mu^2 T + \frac{3\sqrt{3}\pi N_c^2 \gamma}{8} \mu T^2 + \mathcal{O}(T/\mu)^3 \quad (17)$$

The low-temperature leading contribution in (17) is linear in the temperature. Notice that this is common to all RN AdS black hole independent of dimension, which is very similar to the specific heat observed in ordinary Fermi liquids. Note that

$$\frac{N_c^2 \gamma^2}{4} \mu^2 T = \frac{3}{\gamma^2} \cdot \left(\frac{\pi^2}{2} \frac{nT}{\mu} \right) \quad (18)$$

where the bracket is suggestive of the Fermi liquid density. We recall that in the presence of a Fermi surface, the specific heat feeds only from the fermions on the surface fluctuating in its orthogonal direction, thus $\mu^2 T$.

The ratios of the energy and entropy densities to the fermion number density are respectively

$$\frac{\epsilon}{n} = \frac{\pi^2 T^2}{16\gamma^2 \mu} \left(\left(1 + \sqrt{1 + \frac{4\gamma^2 \mu^2}{3\pi^2 T^2}} \right)^2 + \frac{8\gamma^2 \mu^2}{3\pi^2 T^2} \right) \quad (19)$$

and

$$\frac{s}{n} = \frac{\pi^2 T}{2\gamma^2 \mu} \left(1 + \sqrt{1 + \frac{4\gamma^2 \mu^2}{3\pi^2 T^2}} \right) \quad (20)$$

Their small temperature expansions are respectively

$$\frac{\epsilon}{n\mu} = \frac{3}{4} + \sqrt{\frac{3}{4}} \frac{\pi T}{2\gamma\mu} + \mathcal{O}((T/\mu)^2) \quad (21)$$

and

$$\frac{s}{n} = \frac{\pi}{\sqrt{3}} \frac{1}{\gamma} + \frac{\pi^2}{2\gamma^2} \frac{T}{\mu} + \mathcal{O}((T/\mu)^2) \quad (22)$$

Here we comment on an interesting numerology which we do not understand fully. First, we note that at zero temperature and finite density,

$$\frac{\epsilon}{n\mu} = \frac{E}{E + PV} = \frac{E}{E + E/3} = \frac{3}{4} \quad (23)$$

in general thanks to the first law of thermodynamics ($E + PV = TS + \mu N$) and conformal

symmetry ($PV = E/3$). Similarly, at zero density

$$\frac{\epsilon}{TS} = \frac{E}{E + PV} = \frac{E}{E + E/3} = \frac{3}{4} \quad (24)$$

for exactly the same reasons. Now, we remind the readers that at high temperature the *interacting* ϵ and p are also 3/4 the leading black-body contributions [18]. The same factor of 3/4 will also emerge from harmonically trapped cold atoms as we explain below. It is also interesting to notice that the leading density (μ dependent) contributions to n, ϵ, p at high temperature are 3 times the expected scalar and bifundamental fermion contributions.

Finally, the finite density construction followed in [9] and the present work treats the density and temperature on equal footings by means of the RN-AdS black-hole. It is worthy to notice that the RN-AdS black hole horizon

$$r_+ = \frac{l^2}{2} \left(\pi T + \sqrt{\pi^2 T^2 + \frac{4\gamma^2 \mu^2}{3}} \right) \rightarrow \frac{l^2}{\sqrt{3}} \gamma \mu \quad (25)$$

does not vanish at zero temperature. This radius is to be compared with the one at finite temperature and zero density where $r_+ \rightarrow l^2 \pi T$. Interestingly enough, the square root dependence on T, μ in (25) is the one expected from QCD-like arguments [19]. This point will be explored in a further investigation. Here we see that the transition from dense to hot occurs for

$$\pi T \approx 2\mu\gamma/\sqrt{3} \quad (26)$$

One intriguing part of the black-hole holography for cold fermions is the appearance of a finite *entropy* per particle in the zero temperature limit tied with the finite horizon radius. Namely,

$$\frac{S}{N} = \frac{s}{n} = \frac{\pi}{\sqrt{3}\gamma} = f. \quad (27)$$

This persistent cold entropy is at the origin of the viscosity of the superfluid mode in the cold atoms to be detailed below.

What about the third law? The Nernst theorem or the third law of the thermodynamics asserts that the entropy at zero temperature approaches to the minimum value. The minimum value is almost always zero, however, it is not zero if the ground state is degenerate. The holographic approach to strongly coupled fermion systems suggests that the cold atoms have small but measurable entropy at zero temperature. If true, it provides an example of non-vanishing residual entropy. Below we explore the possible measurability of this en-

tropy in cold fermionic atoms either through calorimetry or viscosity measurements in the nonsuperfluid phase near zero temperature.

4 Transport Analysis

Transport in cold holographic fermionic systems has been discussed recently in D4/D8 [22] and D3/D7 [21]. In this section we focus on the fermionic response function for the RN-AdS black-hole construction. Let $\mathbf{A}_\mu(x)$ be the source of the *fermion* bilinear 4-vector current in the boundary of AdS_5 ($r = \infty$). The expectation value of the fermion current is

$$\mathbf{J}_\mu(x) = -i \int d^4y \langle J_\mu(x) J_\nu(y) \rangle_R \mathbf{A}^\nu(y) \quad (28)$$

in the linear response approximation. Here R refers to the retarded correlation function in the state of finite temperature and density. In Fourier space, (28) simplifies

$$\mathbf{J}_\mu(k) = G_{\mu\nu}^R(k) \mathbf{A}^\nu(-k) \quad (29)$$

with the retarded Green's function

$$G_{\mu\nu}^R(k) = -i \int d^4x e^{ik \cdot x} \langle J_\mu(x) J_\nu(0) \rangle_R \quad (30)$$

For a plane-wave baryonic 4-vector field $\mathbf{A} = (0, A^1, A^2, A^3)$ with wavenumber $k^\mu = (\omega, 0, 0, k)$ along the z-direction, the induced current in (29) can be decomposed into longitudinal (along k) and transverse (orthogonal to k) components

$$\begin{aligned} \mathbf{J}_L(k) &= \left(\frac{G_{zz}(k)}{-i\omega} \right) \mathbf{E}_L(-k) \\ \mathbf{J}_T(k) &= \left(\frac{G_{xx}(k)}{-i\omega} \right) \mathbf{E}_T(-k) \end{aligned} \quad (31)$$

with the baryon electric field $\mathbf{E}_{L,T}(-k) = -i\omega \mathbf{A}_{L,T}(-k)$. (31) is just Ohm's law where the longitudinal and transverse conductivities at finite ω , k are respectively

$$\begin{aligned} \sigma_L(k) &= \frac{G_L(k)}{-i\omega} \\ \sigma_T(k) &= \frac{G_T(k)}{-i\omega} \end{aligned} \quad (32)$$

For the RN-AdS black-hole, $G_L = G_{zz}$ and $G_T = G_{xx}$ in the hydrodynamic limit $\omega/T \ll 1$, $k/T \ll 1$ have been explicitly worked out in [11]. We quote the results here for a physical interpretation to follow for a cold fermionic system. In particular

$$\begin{aligned}\sigma_L(k) &= i\omega \left(\frac{A_T}{\omega^2 - k^2/3} + \frac{A_L}{i\omega - D_L k^2} \right) \\ \sigma_T(k) &= - \left(\frac{A_T}{i\omega - D_T k^2} - A_L \right)\end{aligned}\tag{33}$$

with the real residues

$$\begin{aligned}A_L &= \frac{N_c^2 \gamma^2}{4\pi^2} \frac{1}{8b} \left(\frac{2-a}{1+a} \right)^2 \\ A_T &= \frac{N_c^2 \gamma^2}{4\pi^2} \frac{3a}{4b^2} \frac{1}{1+a}\end{aligned}\tag{34}$$

The transverse and longitudinal diffusion constants are tied $D_L/D_T = (2+a)$ and $2D_T = b/(1+a)$. The parameters a and b have been defined in section 2. They are explicitly given as

$$\begin{aligned}a &= \frac{8}{3} \frac{\gamma^2 \mu^2}{\pi^2 T^2} \left(1 + \sqrt{1 + \frac{4\gamma^2 \mu^2}{3\pi^2 T^2}} \right)^{-2} \\ b &= \frac{1}{\pi T} \left(1 + \sqrt{1 + \frac{4\gamma^2 \mu^2}{3\pi^2 T^2}} \right)^{-1}\end{aligned}\tag{35}$$

The elastic mode in(33) follows from fermion number conservation, since $\partial^\mu J_\mu = 0$. Thus $J_0 = (k/\omega)J_z$ so that $G_{00} = (k^2/\omega^2)G_{zz}$. G_{00} triggers scalar sound waves. At zero density or $\mu = 0$, $a = 0$ and the residue $A_T = 0$, hence the elastic or sound mode drops from the longitudinal conductivity. As a result, the DC conductivities for $k = 0$, $\omega \rightarrow 0$ are

$$\sigma(0)_L = \sigma_T(0) = A_L = N_c^2 \gamma^2 T / 4\pi.$$

In contrast, at zero temperature $a = 2$ and the residue $A_L = 0$. As a result the fermionic

conductivities are

$$\sigma_L(0) = \sigma_T(0) = i \frac{A_T}{\omega} = i \frac{N_c^2 \gamma^4 \mu^2}{6\pi^2 \omega} = \frac{n}{\mu - i\omega} \quad (36)$$

The fermion current is $\pi/2$ out of phase with the fermionic electric field. (36) is the Drude conductivity $\sigma_D = ne_F^2/m\tau$ for *free fermions* with an *effective* mass of order μ . The collision time is $\tau \approx 1/\omega$ to this order of the approximation. Indeed, for a harmonic fermion electric field, the free fermion velocity increases linearly with time until the harmonically oscillating field reverses (ballistic regime). This system can be characterized by an electric-like or screening mass

$$m_E^2 = \Pi_{00}(0, \vec{0}) = 3\Pi_L(0, \vec{0}) = \frac{3n}{\mu} = \frac{\partial n}{\partial \mu} \quad (37)$$

where the limits in the first two equalities are $\vec{k} \rightarrow \vec{0}$ and $\omega \rightarrow 0$ sequentially. Here we have defined $\Pi_{\mu\nu} = G_{\mu\nu}/(-i\omega)$.

In general, near the cold fermion limit

$$\begin{aligned} A_L &= \frac{N_c}{12\sqrt{3}\gamma} \frac{T^2}{\mu} + \mathcal{O}(T^3) \\ A_T &= \frac{N_c^2 \gamma^4 \mu^2}{6\pi^2} + \mathcal{O}(T) \\ D_L &= \frac{1}{\sqrt{3}\gamma} \frac{1}{\mu} + \mathcal{O}(T) \\ D_T &= \frac{1}{4} \frac{1}{\sqrt{3}\gamma} \frac{1}{\mu} + \mathcal{O}(T) \end{aligned} \quad (38)$$

The longitudinal diffusive pole vanishes. For cold fermions in RN-AdS the longitudinal channel exhibits a sound mode that propagates with the speed $1/\sqrt{3}$ which is expected in the conformal limit. This point is analogous to the zero sound mode suggested recently in [21, 22] using D3/D7 in the probe approximation. The sound mode is damped at zero temperature because of the persisting finite black hole entropy or degeneracy of the colorless fermionic clusters as we detail below for our case.

In case there is a pairing mechanism, we expect the degenerate fermi liquid to be replaced by a dual repulsive bosonic liquid. The longitudinal sound wave then should be interpreted as the superfluid wave, the zero sound. The fermionic conductivities are replaced by the Meissner-like mass m_M^2 , as the electric-like mass m_E^2 is common to both liquids. These

masses follow readily from the constant mode effective action as well. That is by promoting the zero temperature pressure p to a Lagrange density,

$$\mathcal{L}(A_0, \vec{A}) = \frac{\gamma^2 N_c^2}{24\pi^2} \left((\mu + A_0)^2 - \vec{A}^2 \right)^2 \quad (39)$$

One can read off

$$m_E^2 = 3m_M^2 = \frac{3n}{\mu} \quad (40)$$

(39) was already foreseen in [23] by noting that $m_E^2 = \partial^2 p / \partial \mu^2$ even at finite temperature. The superfluid mode velocity is just $1/\sqrt{3}$ through the longitudinal substitution $A_\mu \rightarrow \partial_\mu \pi$ as a Goldstone mode. The massless superfluid phonons in 3 dimensions are expected to contribute cubically to the specific heat, which is accounted for in (17) at next to next to leading order.

5 Visco-Elastic Analysis

In a cold fermionic system the fermionic conductivities can be analyzed using the visco-elastic equations which are generalized elasticity equations in a medium that is characterized by bulk K and shear M moduli, and bulk ξ and shear η viscosities [22]. Since the cold fermionic response in holography shows no sign of elasticity in the transverse channel, ie σ_T is purely diffusive, it follows that in holography $M = 0$. So the cold fermionic limit is liquid. The bulk viscosity $\xi = 0$ since the underlying fermionic interactions are mediated by a conformal gauge theory. Thus, the cold fermionic liquid in holography is characterized only by a bulk modulus K and a shear viscosity η .

In the visco-elastic framework, we expect [22]

$$\begin{aligned} \tilde{\sigma}_L(k) &= i\omega \left(\frac{n/m}{\omega^2 - (K/mn)k^2 + i(4\eta/3mn)\omega k^2} \right) \\ \tilde{\sigma}_T(k) &= - \left(\frac{n/m}{i\omega - (\eta/mn)k^2} \right) \end{aligned} \quad (41)$$

for massive fermions of mass m and density n with $M = 0$ and $\xi = 0$. These constitutive equations are to be compared with (33) for $a = 2$ or $A_L = 0$

$$\begin{aligned}\sigma_L(k) &= i\omega \left(\frac{A_T}{\omega^2 - k^2/3} \right) \\ \sigma_T(k) &= - \left(\frac{A_T}{i\omega - D_T k^2} \right)\end{aligned}\tag{42}$$

Thus, the identification

$$\begin{aligned}n/m &= A_T \\ K/mn &= c_L^2 = 1/3 \\ \eta/mn &= D_T\end{aligned}\tag{43}$$

The absence of attenuation in the longitudinal conductivity in (42) is a higher order effect not retained in the analysis in [9]. The constitutive or visco-elastic equations show that this calculation is actually not needed. The superfluid mode in (33) and (42) is attenuated through the dispersion law

$$\omega \approx c_L k - i \frac{2D_T}{3} k^2\tag{44}$$

with $c_L = 1/\sqrt{3}$. This is indeed expected since $c_L^2 = \partial p / \partial \epsilon = 1/3$ since $p = \epsilon/3$ in black-hole holography. Also, it follows that

$$\frac{\eta}{n} = n \frac{D_T}{A_T} = \frac{1}{4\sqrt{3}\gamma}\tag{45}$$

after using $m = n/A_T = 2\mu/3$ from (43). This result generalizes readily to arbitrary temperature in holography by using the general expressions for D_T and A_T . Specifically,

$$\frac{\eta}{n} = \frac{\pi T}{8\gamma^2 \mu} \left(1 + \sqrt{1 + \frac{4\gamma^2 \mu^2}{3\pi^2 T^2}} \right)\tag{46}$$

which is in agreement with the thermodynamical relation (22)

$$\frac{\eta}{n} = \frac{\eta}{s} \frac{s}{n} = \frac{1}{4\pi} \frac{s}{n}\tag{47}$$

The ratio $\eta/s = 1/4\pi$ holds for arbitrary T, μ for the RN AdS black-hole thanks to the finite entropy density at $T = 0$. The non-vanishing of the entropy at $T = 0$ provides interesting

example of residual entropy as we noted earlier. This point will be stressed further below.

The emerging physical picture suggested by thermodynamics of cold and strongly coupled fermions in holography is the following: At zero temperature fermions carry finite energy and internal degeneracy f per particle. They form a degenerate liquid (maybe a glassy liquid) with a massless sound mode that damps because of the finite shear viscosity of the liquid caused by f . The latter is due to a finite entropy per particle carried by the clusters as encoded by the nonzero black hole entropy at zero temperature. Away from zero temperature, the clusters break into their constituent quarks that contribute linearly to the specific heat.

6 Trapped Massive Fermions as Massless Fermions

In this paper we wish to extract some generic ratios that we deem general for a strongly coupled fermionic system from RN AdS. Such systems have been studied intensively in the laboratory using optical and magneto-optical traps of Li^6 at Duke [24] and K^{40} at JILA [25]. By sweeping magnetically through the traps, the fermionic atoms are driven through the Feshbach resonance leading to a large S-wave scattering length and strong pair interactions (unitarity limit).

Here we would like to show that a large sample of trapped massive fermions in a harmonic well behave similarly to free massless and untrapped fermions, albeit with higher degeneracy. Indeed, consider free massive fermions in a harmonic trap at finite temperature and density. For simplicity of the argument in this section, we choose units with $m = T = \omega = 1$. Here ω is the frequency of a symmetric trap. For a homogeneous trap and a large density of fermions the Thomas-Fermi approximation applies. In particular, the number N of fermions in the trap is

$$N = \int d\vec{x} n(\vec{x}) = d \int d\vec{x} d\vec{p} \frac{1}{z^{-1} e^{\vec{p}^2/2 + \vec{x}^2/2} + 1} \quad (48)$$

with $z = e^\mu$ the fermionic fugacity. (48) follows from the Thomas-Fermi approximation through the substitution $\mu \rightarrow \mu - V(\vec{x})$ with $V(\vec{x}) = \vec{x}^2/2$ the harmonic potential. Straight-forward algebra yields

$$N = \frac{4dV_5}{V_2} \int d\vec{p} \frac{1}{z^{-1} e^{|\vec{p}|} + 1} \quad (49)$$

While (48) describes massive fermions in a harmonic trap of degeneracy d , its analogue (49)

describes massless and untrapped or free fermions of degeneracy $4dV_5/V_2 > d$. Here V_N is the volume of the S^N unit sphere. This relation holds for arbitrary space dimensions. We note that the degeneracy is T and m dependent when we restore the scales, but that does not affect the Gibbs relations since in the Thomas-Fermi approximation all bulk thermodynamics follows from relations of the type (48).

It follows immediately that the energy per particle and entropy per particle of trapped but free massive fermions and untrapped massless fermions are identical in 3 space dimension for a sufficiently large and homogeneous number of particles to allow for the Thomas-Fermi approximation. In particular,

$$\begin{aligned}\frac{E}{NT} &= 3 \frac{f_4(z)}{f_3(z)} \\ \frac{S}{N} &= 4 \left(\frac{f_4(z)}{f_3(z)} - \frac{1}{4} \ln z \right)\end{aligned}\tag{50}$$

after we have restored T for dimensions with

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty dx \frac{x^{n-1}}{z^{-1}e^x + 1}\tag{51}$$

In the cold limit

$$\begin{aligned}\frac{E}{N\mu} &= \frac{3}{4} + \frac{\pi^2}{2} \left(\frac{T}{\mu} \right)^2 + \mathcal{O}(T^3) \\ \frac{S}{N} &= \pi^2 \left(\frac{T}{\mu} \right) + \mathcal{O}(T^2)\end{aligned}\tag{52}$$

for *both* untrapped massless and harmonically trapped massive fermions. The occurrence of $3/4$ in the massive case for the ratio $E/N\mu$ already signals the conformal transmutation due to the harmonic well in the Thomas-Fermi approximation. This point is made clear in (23). Note that by considering ratios in (50) all the degeneracy mismatch between massless and massive got absorbed.

This remarkable thermodynamical transmutation between free massless and massive but harmonically trapped fermions, allow us to suggest that the present holographic approach with untrapped and massless fermions at strong coupling, is actually well suited for describing trapped fermionic atoms in the unitarity limit. In particular ratios of the type (50) and their expanded version (52) may capture the essential model independent physics. We now proceed

to compare our holographic results with current atomic experiments.

7 Atomic data from Holography

Since we found that *free* cold trapped atoms behave like massless fermions, we are tempted to map the thermodynamics of the *interacting* cold trapped atoms to that of the RN AdS black hole which is relevant for massless and strongly interacting massless fermions. To compare the black hole results with those from atomic experiments, we introduce an interaction parameter ξ

$$\sqrt{\xi} = \mu/k_F = \mu/T_F \quad (53)$$

to characterize the degree of interaction in the ground state. Here, $T_F = E_F = k_F$ is the Fermi temperature for non-interacting massless fermions. The parameter ξ is used for cold fermionic systems but we want to define it in general for strongly interacting fermions like our dual RN AdS black hole with bulk filling branes. For that we recall that for strongly interacting fermions, the energy per particle normalized to the Fermi momentum is defined as

$$\frac{E}{NE_F} = \frac{\epsilon}{n\mu} \frac{\mu}{k_F} = \frac{3}{4} \sqrt{\xi} \quad (54)$$

To determine ξ in terms of the gravity and brane parameters, we note that the fermion number is unaffected by the interaction. Consider first the noninteracting fermion system, where the density is

$$n_{free} = dk_F^3/6\pi^2 \quad \text{with} \quad d = N_f N_c \quad (55)$$

for spinless fermions with N_f flavor (in the fundamental representation of color and flavor). As we turn on the strong (attractive) interaction, we expect the fermions to bind into composites and the Fermi energy μ to depart from its free value k_F . The number density of the interacting system is given in our holographic model by

$$n_{int} = \frac{N_c^2 \gamma^4}{6\pi^2} \mu^3. \quad (56)$$

We take the parameter determined from the $D3/D7$ system to set $\gamma = (N_f/N_c)^{1/2}$. Since fermion number is unchanged, we find that the fundamental parameter γ in holography ties

with the fundamental parameter ξ in cold fermion systems through ²

$$\xi = \gamma^{-4/3} \quad (57)$$

Now we want to map the thermodynamics of a cold fermion system to that of the charges in bulk filling branes discussed in section 3 and 4. Atomic experiments in the unitarity limit give $\xi = 0.41$ [26, 27] to fit the ground state energy. We will use this empirical value in our numerical estimates below.

The entropy per particle in holography is nonzero at the coldest point

$$\frac{S}{N} = \frac{\pi}{\sqrt{3}} \xi^{3/4} \approx 0.93. \quad (58)$$

Recent atomic physics experiments [26] have measured the entropy of trapped cold atoms by using magnetic field sweeps across the trap. The reported measurements reveal a coldest entropy per particle of about 1 that is consistent with our estimate. The entropy (58) is at the origin of a finite cold viscosity per particle

$$\frac{\eta}{n} = \frac{\xi^{3/4}}{4\sqrt{3}} \approx 0.07 \quad (59)$$

which we note to be larger than $1/6\pi$ as conjectured in [4]. At this point, it is worth noting that the atomic measurements of $\eta/n = \langle \alpha \rangle$ in [26] at their coldest point are also consistent with our holographic estimate. If one extrapolates the entropy to be zero at zero temperature, it suggests $\eta \sim s/4\pi \sim 0$ in the ground state in the unitarity limit, which is not consistent with $\eta/s = 1/4\pi$. Current atomic measurements of the entropy and/or viscosity at the coldest point in the unitarity limit offers a direct measurement of the cold black hole entropy through holography. At finite temperature, the entropy per particle is

$$\frac{S}{N} = \frac{\pi^2}{2} \xi x \left(1 + \sqrt{1 + \frac{4\xi^{-1/2}}{3\pi^2 x^2}} \right) \quad (60)$$

with $x = T/T_F$. At high temperature the entropy per particle is $S/N \approx \pi^2 x$. Similarly, the energy per particle is

$$\frac{E}{NT_F} = \frac{\pi^2}{16} \xi x^2 \left(\left(1 + \sqrt{1 + \frac{4\xi^{-1/2}}{3\pi^2 x^2}} \right)^2 + \frac{8\xi^{-1/2}}{3\pi^2 x^2} \right) \quad (61)$$

²We note that this is the only time where we make explicit use of the parameters set of the D3/D7 embedding in [9]. Without it, $\xi = (d/N_c^2 \gamma^4)^{2/3}$.

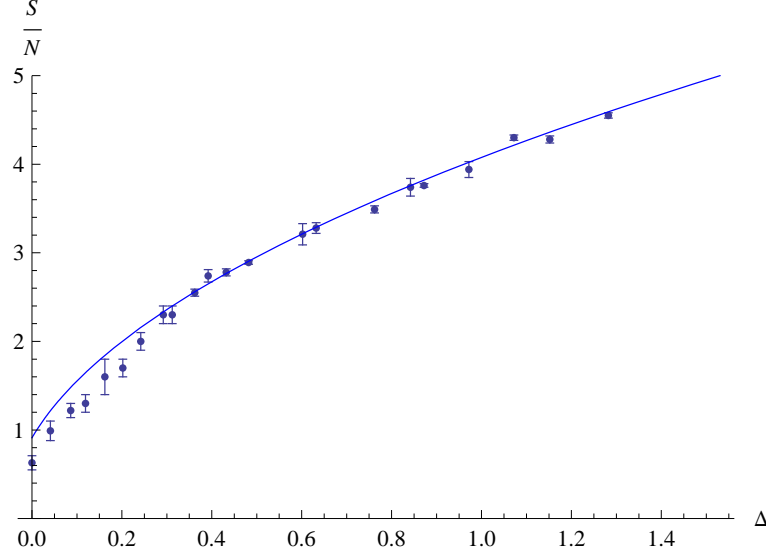


Figure 1: S/N versus energy Δ : Holography (solid line: $\xi = 0.4$) and atomic data [26].

To make the comparison with atomic experiments manifest, we set E_0 to be the value of E at zero temperature or $x = 0$, and defined the shifted energy $\Delta = (E - E_0)/NT_F$ which is zero for $x = 0$. Thus

$$\Delta = \frac{E - E_0}{NT_F} = \frac{\pi^2}{16} \xi x^2 \left(1 + \sqrt{1 + \frac{4\xi^{-1/2}}{3\pi^2 x^2}} \right)^2 - \frac{\xi^{1/2}}{12} \quad (62)$$

Δ is the analogue of $(E_{840} - E_0)/E_F$ used in the empirical analysis in [26].³ This translates to

$$\Delta \approx 0.51x \left(x + (x^2 + 0.21)^{0.5} \right) \quad (63)$$

which is to be compared with $\Delta \approx 2.64 x^{1.43}$ empirically in the nonsuperfluid phase with $x > T/T_F \approx 0.34$ [26].

In terms of the shifted energy (62), the entropy per particle simplifies

$$\frac{S}{N} = (2\pi\xi^{1/2}) \left(\Delta + \frac{\xi^{1/2}}{12} \right)^{1/2} \approx 4.02 (\Delta + 0.05)^{0.5} \quad (64)$$

which is to be compared with

³840 here is the magnetic field in Gauss unit to confine the fermions.

$$\frac{S}{N} = (4.0 \pm 0.2) \Delta^{0.45 \pm 0.01} \quad (65)$$

obtained empirically in [26, 27] in the nonsuperfluid regime or $E/T_F > 0.94 \pm 0.05$. The holographic energy dependence of the entropy compares remarkably with the atomic experiments as we show in Fig. 1. We note that the small temperature corrections in (52) for free massless fermions give at low temperature $S/N \approx \pi\sqrt{2\Delta} \approx 4.44 \Delta^{0.5}$ which overshoots the data. The corresponding viscosity per particle reads

$$\frac{\eta}{n} \approx 0.32 (\Delta + 0.05)^{0.5} \quad (66)$$

8 Conclusions

We have used an holographic construction to analyze cold and hot massless fermions including fermionic back-reaction as discussed recently by one of us [9]. The ensuing gravitational metric is RN AdS black-hole as opposed to AdS black hole for the probe approximation. Holography allows a number of predictions for the bulk thermodynamics and transport near the cold point, that are readily amenable to testing against current atomic experiments involving Li^6 or K^{40} atoms near the so called Feshbach resonance.

We have shown that holography supports a finite entropy and thus shear viscosity at the coldest point. While the D3/D7 embedding allows for a derivation of the fermionic ground state energy per particle, the result depends only on $\sqrt{\xi} = (N_c/N_f)^{1/3}$, a fundamental parameter in holography. We recall that trapped atomic experiments put $\xi = 0.41$. We have found that the effects of temperature as suggested by the RN AdS black hole construction compare favorably with current atomic measurements. In particular the entropy versus energy dependence in cold atoms is derived accurately. Given the simplicity of the theoretical construction this is certainly very pleasing.

The idea of a minimal cold viscosity per particle in trapped atoms was initially suggested in [4] who argued that $\eta/n \geq 1/6\pi \approx 0.05$. It also follows from a quasi-normal mode analysis using D3/D7 in the probe approximation [21, 22]. In particular $\eta/n = 1/4$ [22]. Our analysis differs from [21, 22] in that the full gravity back-reaction due to the presence of the fermions is considered. Our construction also shows explicitly how the shear viscosity ties with the entropy (degeneracy) of the cold black hole at zero temperature.

The specific heat and equation of state $p = \epsilon/3$ we discussed in this paper strongly suggest that the relevant degree of freedom at low temperature is massless and fermionic.

Very recently another bosonic system is reported to show the thermodynamic and transport behavior of fermi liquid [28]. However, the conductivity of charged ads BH at a cold point is $\sim T^2$ while we expect $1/T^2$ for fermi liquid. Therefore the issue of gravity dual of fermionic system is not totally clear and it is still waiting further investigation.

The existence of a finite entropy at zero temperature for charged black holes is very intriguing as no Bekenstein-Hawking radiation can be assigned to it in standard gravity. Precise measurements of the entropy of trapped cold atoms may shed light on this fundamental concept at the coldest point. The latter is currently probed only by theoretical extrapolations to have zero entropy at $T = 0$. We have shown that these extrapolations conflict with the finite shear viscosity quoted in [26] at the coldest point.

Note added: After submitting the first version of this paper to the archive, Ref. [29] was brought to our attention regarding some overlap with sections 3,4. However, our results are different.

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References

- [1] K. M. O’Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade and J. E. Thomas, “Observation Of A Strongly Interacting Degenerate Fermi Gas Of Atoms,” *Science* **298**, 2179 (2002).
- [2] H. Heiselberg, “Fermi systems with long scattering lengths,” *Phys. Rev. A* **63**, 043606 (2002) [arXiv:cond-mat/0002056]; J. Carlson, S. Y. Chang, V. R. Pandharipande and K. E. Schmidt, “Superfluid Fermi Gases with Large Scattering Length,” *Phys. Rev. Lett.* **91**, 050401 (2003); T. L. Ho, “Universal Thermodynamics of Degener-

- ate Quantum Gases in the Unitarity Limit,” Phys. Rev. Lett. **92**, 090402 (2004) [arXiv:cond-mat/0309109].
- [3] H. Heiselberg, ”Collective Modes Of Trapped Gases At The Bec-Bcs Crossover”, Phys. Rev. Lett. **93** (2004) 040402
- [4] B. A. Gelman, E. V. Shuryak and I. Zahed, “Ultracold strongly coupled gas: A near-ideal liquid,” Phys. Rev. A **72**, 043601 (2005) [arXiv:nucl-th/0410067].
- [5] P. Kovtun, D. T. Son and A. O. Starinets, “Viscosity in strongly interacting quantum field theories from black hole physics,” Phys. Rev. Lett. **94**, 111601 (2005) [arXiv:hep-th/0405231]; G. Policastro, D. T. Son and A. O. Starinets, “The shear viscosity of strongly coupled $N = 4$ supersymmetric Yang-Mills plasma,” Phys. Rev. Lett. **87**, 081601 (2001) [arXiv:hep-th/0104066].
- [6] L. Luo, B. Clancy, J. Joseph, J. Kinast and J. E. Thomas, “Measurement of the Entropy and Critical Temperature of a Strongly Interacting Fermi Gas,” Phys. Rev. Lett. **98**, 080402 (2007) [arXiv:cond-mat/0611566]; T. Schafer, “The Shear Viscosity to Entropy Density Ratio of Trapped Fermions in the Unitarity Limit,” Phys. Rev. A **76**, 063618 (2007) [arXiv:cond-mat/0701251].
- [7] K. Y. Kim, S. J. Sin and I. Zahed, “Dense hadronic matter in holographic QCD,” arXiv:hep-th/0608046; N. Horigome and Y. Tanii, “Holographic chiral phase transition with chemical potential,” JHEP **0701**, 072 (2007) [arXiv:hep-th/0608198].
- [8] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, Phys. Rev. D **60**, 064018 (1999) [arXiv:hep-th/9902170].
- [9] S. J. Sin, “Gravity Back-reaction to the Baryon Density for Bulk Filling Branes,” JHEP **0710**, 078 (2007) [arXiv:0707.2719 [hep-th]].
- [10] D. T. Son and A. O. Starinets, JHEP **0603**, 052 (2006) [arXiv:hep-th/0601157].
- [11] X. H. Ge, Y. Matsuo, F. W. Shu, S. J. Sin and T. Tsukioka, “Density Dependence of Transport Coefficients from Holographic Hydrodynamics,” Prog. Theor. Phys. **120**, 833 (2008) [arXiv:0806.4460 [hep-th]]; Y. Matsuo, S. J. Sin, S. Takeuchi, T. Tsukioka and C. M. Yoo, “Sound Modes in Holographic Hydrodynamics for Charged AdS Black Hole,” arXiv:0901.0610 [hep-th].

- [12] S. S. Lee, “A Non-Fermi Liquid from a Charged Black Hole: A Critical Fermi Ball,” arXiv:0809.3402 [hep-th].
- [13] H. Liu, J. McGreevy and D. Vegh, “Non-Fermi liquids from holography,” arXiv:0903.2477 [hep-th].
- [14] M. Cubrovic, J. Zaanen and K. Schalm, “Fermions and the AdS/CFT correspondence: quantum phase transitions and the arXiv:0904.1993 [hep-th].
- [15] D. T. Son, Phys. Rev. D **78**, 046003 (2008) [arXiv:0804.3972 [hep-th]].
- [16] A. Adams, K. Balasubramanian and J. McGreevy, JHEP **0811**, 059 (2008) [arXiv:0807.1111 [hep-th]].
- [17] D. V. Deryagin, D. Y. Grigoriev and V. A. Rubakov, “Standing wave ground state in high density, zero temperature QCD at large $N(c)$,” Int. J. Mod. Phys. A **7**, 659 (1992); E. Shuster and D. T. Son, “On finite-density QCD at large $N(c)$,” Nucl. Phys. B **573**, 434 (2000) [arXiv:hep-ph/9905448]; B. Y. Park, M. Rho, A. Wirzba and I. Zahed, “Dense QCD: Overhauser or BCS pairing?,” Phys. Rev. D **62**, 034015 (2000) [arXiv:hep-ph/9910347].
- [18] S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, “Coupling constant dependence in the thermodynamics of $N = 4$ supersymmetric Yang-Mills theory,” Nucl. Phys. B **534**, 202 (1998) [arXiv:hep-th/9805156].
- [19] R. A. Schneider, “The QCD running coupling at finite temperature and density,” arXiv:hep-ph/0303104.
- [20] K. Y. Kim, S. J. Sin and I. Zahed, “The Chiral Model of Sakai-Sugimoto at Finite Baryon Density,” JHEP **0801**, 002 (2008) [arXiv:0708.1469 [hep-th]].
- [21] A. Karch, D. T. Son and A. O. Starinets, “Holographic Quantum Liquid,” Phys. Rev. Lett. **102**, 051602 (2009). M. Kulaxizi and A. Parnachev, “Holographic Responses of Fermion Matter,” Nucl. Phys. B **815**, 125 (2009) [arXiv:0811.2262 [hep-th]].
- [22] K. Y. Kim and I. Zahed, “Baryonic Response of Dense Holographic QCD,” JHEP **0812**, 075 (2008) [arXiv:0811.0184 [hep-th]].
- [23] J. I. Kapusta, “Finite-Temperature Field Theory”, Camb. Univ. Press (1989).

- [24] L. Luo, B. Clancy, J. Joseph, J. Kinast and J. E. Thomas, "Measurement of the Entropy and Critical Temperature of a Strongly Interacting Fermi Gas," Phys. Rev. Lett. **98**, 080402 (2007) [arXiv:cond-mat/0611566].
- [25] J. T. Stewart, J. P. Gaebler, C. A. Regal and D. S. Jin, "The potential energy of a ^{40}K Fermi gas in the BCS-BEC crossover," Phys. Rev. Lett. **97**, 220406 (2006) [arXiv:cond-mat/0607776].
- [26] L. Luo and J. E. Thomas, "Thermodynamics Measurements in a Strongly Interacting Fermi Gas", [arXiv:0811.1159[cond-mat]].
- [27] A. Turpalov, J. Kinast, B. Clancy, L. Luo, J. Joseph and J. E. Thomas, "Is a gas of Strongly Interacting Atomic Fermions a Perfect Fluid?" [arXiv:0707.2574[cond-mat]].
- [28] S. S. Gubser and F. D. Rocha, arXiv:0911.2898 [hep-th].
- [29] Soo-Jong Rey, "String Theory on Thin Semiconductors", Progress of Theoretical Physics Supplement, 177 (2009) 128-142.